JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 17, No. 1, January–February 1994

Computational Results for a Feedback Control for a Rotating Viscoelastic Beam

Athena Makroglou* and Richard K. Miller†
Iowa State University, Ames, Iowa 50011
and
Steven Skaar‡
University of Notre Dame, Notre Dame, Indiana 46556

This paper studies a control system for a one-dimensional, distributed, viscoelastic structure whose constitutive law is modeled using fractional order derivatives. Skaar, Michel, and Miller ("Stability of Viscoelastic Control Systems," *IEEE Transactions*, Vol. AC-33, No. 4, 1988, pp. 348-357) have proposed a modified root locus scheme for such systems and have suggested an approximation method for computation of solutions which relies on Laplace transformations. We show here, via a case study for a slewing beam, how these approximations can be carried out. In Skaar et al., the focus was on determining stability. In contrast, this paper shows how time domain performance can be assessed. We show that the necessary inverse Laplace transforms are obtained as solutions of a weakly singular system of Volterra integral equations. Comparisons are made between the fractional derivative model and a Kelvin-Voight constitutive model.

Nomenclature

$A(t)$	= integral equation kernal function
a, b, E, ν	= viscoelastic model parameters
$E + b(\partial^\nu/\partial t^\nu)$	= strain operator
erfc(t)	= complementary error function
K^*	= feedback gain constant
K_{jk}	= structural stiffness matrix
$K_1(s)$	= feedback gain function
M_{jk}	= structural mass matrix
$R(t)$	= response of the integral equation to unit input
$R_1(t)$	= integral equation resolvent function
s	= Laplace transform variable
t	= time starting at the onset of motion
$w_j(t, p)$	= modal response function
$y(t, z)$	= beam state variable
Γ	= gamma function
$\varphi_j(z)$	= Ritz basis function
$\theta(t)$	= hub pointing angle
θ_d	= desired hub pointing angle
$\partial^\nu/\partial t^\nu$	= fractional derivative of order ν
$1 + a(\partial \nu/\partial t^\nu)$	= stress operator

Introduction

WE consider here the problem of constructing a feedback control system for a one-dimensional, distributed, viscoelastic structure whose constitutive law is modeled using fractional order derivatives. Our starting point is the results in Ref. 1 where the classical root locus method was adapted to such control systems and Ref. 2 where the results of Ref. 1 are generalized. For a discussion of fractional derivative models of viscoelastic structures, see Refs. 3–6. In Refs. 1 and 2 the authors assume that the stress σ and the strain ϵ are related via the constitutive relation

$$\left(1 + a\frac{\partial^\nu}{\partial t^\nu}\right)\sigma = \left(E + b\frac{\partial^\nu}{\partial t^\nu}\right)\epsilon \tag{1}$$

where a, b, and E are positive constants and $0<\nu<1$.

In the analysis of Ref. 1, the authors introduce a Ritz approximation. This approximation is used in the root locus analysis. We will show here that this Ritz approximation can also be used to obtain numerical approximations for time domain control output variables. Thus the same approximations can be used to evaluate the system's performance. We shall analyze in detail a slewing beam problem. This case study will introduce all of the necessary ideas.

Beam Problem

We consider a viscoelastic beam (see Fig. 1) that is free at one end whereas the other end is rigidly attached to a rigid, hinged hub. The problem is to bring the beam to rest, i.e., $y(t, z) \equiv 0$, at a desired pointing angle θ_d. Control is to be achieved by one control torque u on the hub.

We assume that gravity acts into the page (i.e., along the axis of the hub) in a direction perpendicular to the motion of the hub and beam. Hence gravitational effects are neglected. The beam is assumed to have length L, mass per unit length ρ, cross-sectional second moment I about the neutral axis, and

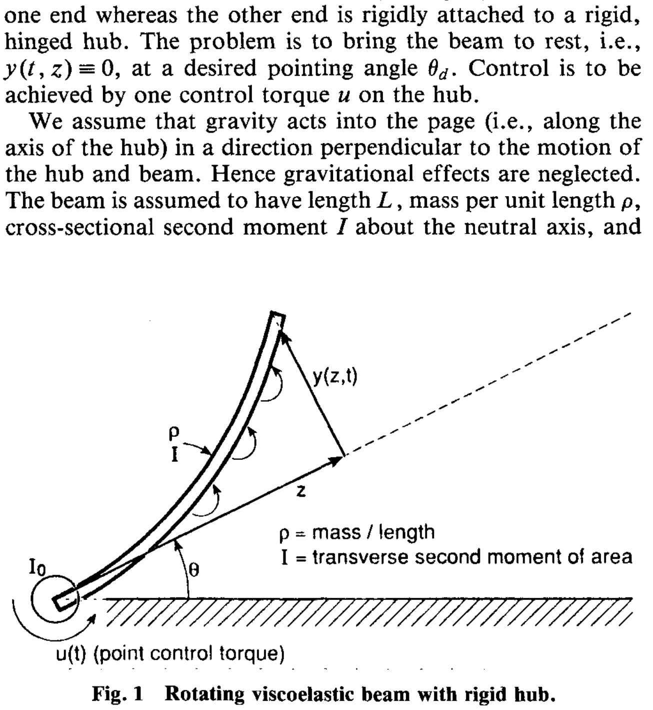

Fig. 1 Rotating viscoelastic beam with rigid hub.

Received July 16, 1992; revision received April 21, 1993; accepted for publication May 8, 1993.

*Visiting Professor, Department of Mathematics.
†Professor, Department of Mathematics.
‡Professor, Department of Aerospace and Mechanical Engineering. Member AIAA.

the constitutive law (1). Let I_0 be the mass moment of inertia of the hub and define

$$I_T = I_0 + \frac{\rho L^3}{3}$$

Define a complex quantity h via the relation

$$h = s\left(\frac{1+as^\nu}{E+bs^\nu}\right)^{1/2} \tag{2}$$

where all roots are principle roots with branch cuts along the negative real axis.

Let $\bar{\theta}(s) = \mathcal{L}\{\theta\}$ be the Laplace transform of the hub pointing angle $\theta(t)$ with $\theta(0)=0$ (i.e., the zero-state pointing angle). As in Ref. 1 we take feedback of the form

$$\bar{u}_1(s) = K_1(s)\left[\theta_d/s - \bar{\theta}(s)\right] \tag{3}$$

where $K_1(s) = K^* s^2/h^2$ and θ_d is the desired pointing angle. The unknown constant gain K^* can be chosen using the root locus technique proposed in Ref. 1. We will use the functions

$$\varphi_j(z) = z^{j+1}, \qquad j = 1, 2, 3, \ldots, p \tag{4}$$

and the approximation

$$y(t,z) = \sum_{j=1}^{p} q_j(t)\varphi_j(z)$$

Then according to Ref. 1 (p. 351) the corresponding approximation for the zero-state hub pointing angle takes the form

$$\bar{\theta}(s) = (\theta_d/s)C(h^2)$$

where $C(h^2) = K_1(s)G_1(s)/[1+K_1(s)G_1(s)]$ and G_1 is obtained as follows:

Define matrices K and M and a vector b by the relations

$$K_{jk} = \int_0^L I\varphi_j''(z)\varphi_k''(z)\,\mathrm{d}z$$

$$M_{jk} = \int_0^L \rho\varphi_j(z)\varphi_k(z)\,\mathrm{d}z$$

$$b_j = \int_0^L \rho\varphi_j(z)\,\mathrm{d}z$$

It was shown in Ref. 1 that

$$G_1(s) = s^{-2}\left[I_T - h^2 b^T (K + h^2 M)^{-1} b\right]^{-1}$$

Notice that $C(h^2)$ will be a function of h^2 only. For convenience of notation let $x = h^2$ so that

$$C(x) = \frac{K^*}{K^* + x\left[I_T - x b^T (K + xM)^{-1} b\right]} = \frac{p(x)}{p_1(x)}$$

where $p(x)$ and $p_1(x)$ are polynomials in x and the degree of $p_1(x)$ is greater than the degree of $p(x)$. If $p_1(x)$ has distinct roots α_i, then there will be constants β_i such that

$$C(x) = \sum_{j=1}^{Q} \frac{\beta_j}{x+\alpha_j}$$

We will choose K^* as in Ref. 1 so that the system is stable. In this case the roots $\alpha_i = p_i^2$ will be positive. Hence

$$C(h^2) = \sum_{j=1}^{Q} \frac{\beta_j}{h^2 + p_j^2}, \qquad h^2 = s^2\left(\frac{1+as^\nu}{E+bs^\nu}\right) \tag{5}$$

is the transfer function that approximately characterizes this feedback system, $c(t) = \mathcal{L}^{-1}\{c[h^2(s)]\}$ is the corresponding impulse response function, and the zero-state angular response

$$\theta(t) = \theta_d \int_0^t c(s)\,\mathrm{d}s$$

is obtained by one integration. Our purpose will be to demonstrate that $\theta(t)$ and $c(t)$ are computable. More details about the approximation and the feedback scheme can be found in Ref. 1.

We remark that the form (5) of the transfer function is not unique to the configuration considered in Fig. 1. Rather, as discussed in Refs. 1 and 2, this form will characterize a wide variety of closed-loop systems composed of rigid and viscoelastic members. The finite-dimensional approximation of the equations of motion is based on a combination of consistent (e.g., finite element) approximation and the correspondence principle. The computation of the transfer function proceeds as follows:

1) The rigid-viscoelastic system is replaced by the corresponding rigid-elastic system and the elastic constant E is set equal to one. Thus, each viscoelastic member is replaced by an elastic equivalent.

2) A consistent approximation is applied to the elastic system in the usual way. The transfer function obtained using this approximation will have the form

$$T(x) = \sum_{j=1}^{N} \frac{\alpha_j}{s^2 + q_j^2}$$

3) Once $T(s)$ has been computed, the transfer function for the viscoelastic system is immediately obtained by replacing s with h [from Eq. (2)] in the formula for $T(s)$. Obviously this transfer function will always have the same form as Eq. (5)!

For small numbers of Ritz functions [i.e., for p small in Eq. (4)] these computations can be carried out exactly using a symbolic manipulation computer program. This was done for $p=2$ and 3 in Eq. (4). The results of the two computations were substatially the same. Since the case $p=3$ is expected to give more accurate approximation, we will take $p=3$ henceforth.

For example purposes we choose $\theta_d = \rho = L = I = I_T = 1$. Hence $I_0 = 2/3$. It can be shown that the gain $K^* = 1$ stabilizes the system. We choose $K^* = 1$ in $K_1(s)$ of Eq. (3).

We choose $\nu = 0.5$, $E = 7.6\times10^5$, $a = 2.95\times10^5$, and $b = 0.001$. These constants model the constitutive law for Butyl B252, a representative viscoelastic material. In this case

$$p(x) = \frac{48}{5} + \frac{418}{528}x + \frac{41}{25{,}200}x^2 + \frac{1}{8{,}890{,}560}x^3$$

and

$$p_1(x) = \frac{48}{5} + \frac{2{,}310{,}698{,}880}{222{,}264{,}000}x + \frac{70{,}639{,}380}{222{,}264{,}000}x^2 + \frac{74701}{222{,}264{,}000}x^3 + \frac{1}{55{,}566{,}000}x^4$$

Hence $Q = 4$ in Eq. (5). The constants β_i and p_i are given in Table 1.

Table 1 Feedback coefficient values

p_j	p_j^2	β_j
0.975212	0.951038	0.903159
5.73930	32.9396	1.57302
31.0359	963.227	2.42009
132.959	17678.0	1.35374

We have reduced the computation of $c(t)$ and $\theta(t)$ to inverting Laplace transforms of functions of the form

$$\bar{w}_1(s,p) = \frac{1}{h^2+p^2} = \frac{E+bs^\nu}{s^2(1+as^\nu)+p^2(E+bs^\nu)} \tag{6}$$

and of the form

$$\bar{w}_2(s,p) = (1/s)\bar{w}_1(s,p) \tag{7}$$

In our special case $\nu = 0.5$, it is known, cf., e.g., Ref. 7 (p. 211) or Ref. 8, that if $F(s) = \mathcal{L}\{f\}$, then

$$\mathcal{L}^{-1}\{F(\sqrt{s})\} = \int_0^\infty \frac{v}{2t\sqrt{\pi t}} e^{-v^2/4t} f(v)\,dv \tag{8}$$

In our case $f(t)$ will be of the form

$$f(t) = \sum_{i=1}^{5} a_i e^{b_i t} \tag{9}$$

where $Re\ b_i \neq 0$ for all i, $b_1 < 0$, $b_2 = \bar{b}_3$, and $b_4 = \bar{b}_5$ ($\bar{b}_j$ means b_j conjugate). For any complex number b

$$\int_0^\infty \frac{v}{2t\sqrt{\pi t}} e^{-v^2/4t} e^{bv}\,dv = \frac{e^{b^2 t}}{\sqrt{\pi t}} \int_0^\infty \frac{v}{2t} e^{-(1/4t)(v-2tb)^2}\,dv$$
$$= \frac{e^{b^2 t}}{\sqrt{\pi t}} \int_{-b\sqrt{t}}^{\infty} (w+\sqrt{t}b)e^{-w^2}\,dw$$

It is known, cf. Ref. 9 (p. 299, line 7.2.3), that this last integral can be written as a complex contour iterated integral

$$i[\mathrm{erfc}(-b\sqrt{t})] = \int_{-b\sqrt{t}}^{\infty} \mathrm{erfc}(v)\,dv$$

Hence Eqs. (8) and (9) yield the exact inverse

$$\mathcal{L}^{-1}\{F(\sqrt{s})\} = \frac{1}{\sqrt{\pi t}} \sum_{i=1}^{5} a_i e^{b_i^2 t} i[\mathrm{erfc}(-b_i\sqrt{t})]$$

However, this result does not appear to be particularly useful for either numerical computations or for finding analytic properties of the solution.

Volterra Integral Equations

Expressions of the form (6) or (7) can in theory be inverted by employing a numerical Laplace transform inverter. For description of such methods see, for example, Refs. 10-15. We have not tested all routines available but numerical results obtained by using one such method, cf. Ref. 15, showed that it is possible that a very large number of function evaluations is required for convergence for certain values of t. There were cases when an upper bound for the number of function evaluations as large as 48,000 was reached without convergence. Hence we desire a second method of computation. We found that the inversion problem can be reformulated as a Volterra integral equation problem. This integral equation problem can be solved numerically. Background material on Volterra integral equations can be found for example in Refs. 16 or 17 and numerical background in Refs. 18-20. The results of this section can be viewed as a special case of the work in Ref. 2. Let $A(t) = b^{-1}t^{\nu-1}/\Gamma(\nu)$ where $b > 0$, $0 < \nu < 1$, and $\Gamma(\nu)$ is the γ function. Let $R_1(t)$ be the resolvent of $A(t)$, i.e., $R_1(t)$ solves

$$R_1(t) = A(t) - \int_0^t A(t-v)R_1(v)\,dv \tag{10}$$

Then $R_1(t)$ will exist for all $t > 0$, it is absolutely integrable on $(0, \infty)$, and it has Laplace transform

$$\bar{R}_1(s) = (1+bs^\nu)^{-1}$$

Moreover, the solution $R(t)$ of the equation

$$R(t) = 1 - \int_0^t A(t-v)R(v)\,dv \tag{11}$$

is

$$R(t) = 1 - \int_0^t R_1(v)\,dv$$

With $R_1(t)$ given above, let $\omega_1(t,p)$ solve

$$w_1''(t,p) = -p^2\left[\frac{a}{b}\,\omega_1(t,p) + \left(E - \frac{a}{b}\right)\int_0^t R_1(t-v)\omega_1(v,p)\,dv\right] + \beta\left(E-\frac{a}{b}\right)R_1(t) \tag{12}$$

with $\omega_1(0,p) = 0$ and $\omega_1'(0,p) = \beta$. Here $\beta = \beta_j$ and $p = p_j$ will be taken from Table 1. Lengthy but elementary computations show that $\omega_1(t,p)$ has Laplace transform

$$\bar{w}_1(s,p) = \frac{\beta}{s^2[(1+as^\nu)/(E+bs^\nu)]+p^2} = \frac{\beta}{h^2+p^2}$$

This can be turned around into the inversion formula

$$\mathcal{L}^{-1}\{\beta/(h^2+p^2)\} = \omega_1(t,p)$$

Inverting Eq. (6) is therefore equivalent to solving the two integral equations (10) and (12).

For numerical purposes we shall use integrated forms of Eqs. (10) and (12). Hence we integrate Eqs. (12) over $[0, t]$. The result is the system

$$w_1'(t,p) = -p^2\left[E\int_0^t w_1(v,p)\,dv + \left(\frac{a}{b}-E\right)\int_0^t R(t-v)w_1(v,p)\,dv\right] + \beta\left[E - \left(E-\frac{a}{b}\right)R(t)\right] \tag{13}$$

$$R(t) = 1 - \int_0^t \frac{1}{b\Gamma(\nu)}(t-v)^{\nu-1}R(v)\,dv \tag{14}$$

with $w_1(0,p) = 0$. The inverse Laplace transform of Eq. (7) can be computed as

$$w_2(t,p) = \int_0^t w_1(v,p)\,dv \tag{15}$$

or we can solve for $w_2(t,p)$ using the system consisting of Eq. (14) plus the equation

$$w_2'(t,p) = -p^2\left[E\int_0^t w_2(s,p)\,ds + \left(\frac{a}{b}-E\right)\int_0^t R(t-v)w_2(v,p)\,dv\right] + \beta\left[Et - \left(E-\frac{a}{b}\right)\int_0^t R(v)\,dv\right]$$

We view Eqs. (13) and (14) as a system of two coupled equations for the two unknowns $w_1(t,p)$ and $R(t)$. Actually $R(t)$ can be found analytically. Indeed

$$R(t) = 1 + \sum_{n=1}^{\infty} \frac{1}{\Gamma(n\nu+1)}\left(-\frac{t^\nu}{b}\right)^n, \qquad t>0 \tag{16}$$

for any $\nu \in (0, 1)$. Hence $R(t)$ is a Mittag-Leffler function of order ν, cf. Ref. 2 or Ref. 21 (p.196), for more details. In the special case $\nu = 0.5$, it can also be shown that

$$R(t) = \frac{2e^{t/b^2}}{\sqrt{\pi}} \int_{\sqrt{t}/b}^{\infty} e^{-r^2}\,dr \tag{17}$$

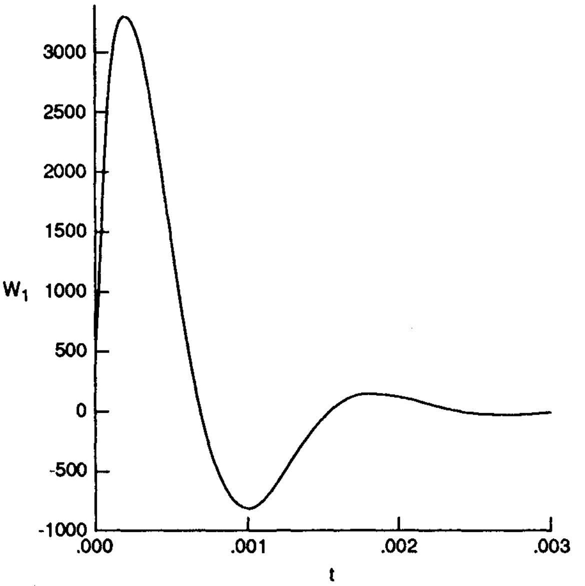

Fig. 2 Butyl B252 beam, $p_1 = 0.975212$, $w_1(t, p_1)$.

We shall comment further on the use of these formulas in the fourth section.

Numerical Results

Numerical results for the control system were obtained by solving system (13) and (14) with relevant values of p. This system consists of one Volterra integro-differential equation and one weakly singular integral equation in the two unknowns $w_1(t, p)$ and $R(t)$. The methods used are the so-called block-by-block methods combined with product integration on graded meshes. It is planned that the numerical procedures will be discussed in more detail later. For now we note that block-by-block methods are implicit Runge-Kutta methods that were first applied to Volterra integral equations by De Hoog and Weiss.[22] They were extended to the solution of Volterra intego-differential equations with smooth kernels by Makroglou[23,24] and Brunner[25] and to integro-differential equations with weakly singular kernels by Makroglou[26–28] and Brunner.[29,30]

Numerical results (see Figs. 2–7) were obtained for all values of p_j in Table 1. We used $\beta = 1$ without loss of generality. Very small step sizes had to be used to capture the highly oscillatory

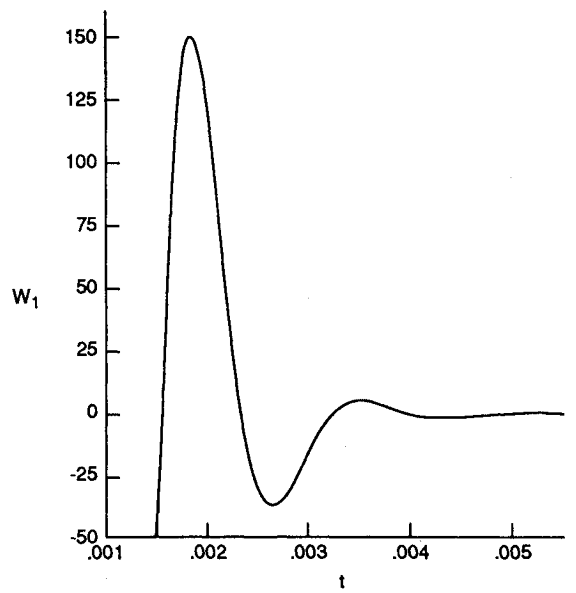

Fig. 3 $w_1(t, p_1)$ detail.

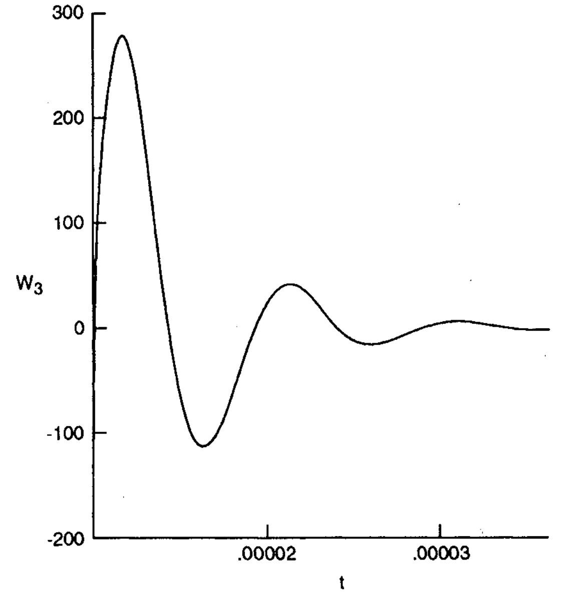

Fig. 5 $p_3 = 31.035898$, $w_1(t, p_3)$.

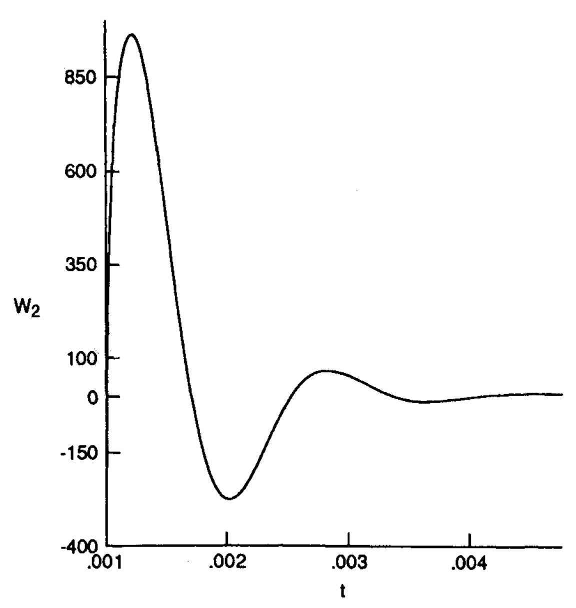

Fig. 4 $p_2 = 5.739303$, $w_1(t, p_2)$.

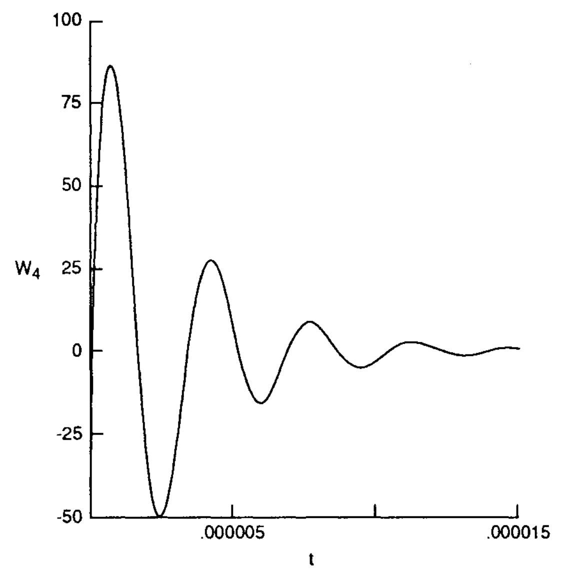

Fig. 6 $w_1(t, p_4)$, $p_4 = 132.958633$.

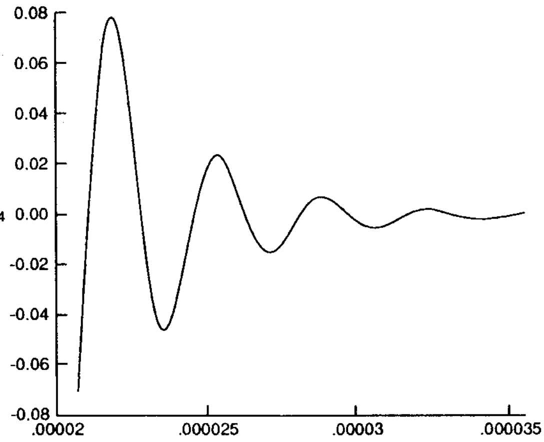

Fig. 7 $w_1(t, p_4)$ detail.

nature of the solutions. Integration was carried out until the solution settled to within 0.1 of its limiting value. The results obtained by solving Eqs. (13) and (14) were verified, for all p_j, using the numerical Laplace transform inverter ACM-619 by Piessens and Huysmans.[15] The results for the two methods agree to at least two decimal places everywhere with better agreement at the beginning of the interval of integration.

Computations were also done using formulas (15) and (16) to obtain the values of $R(t)$ required for the numerical solution of Eqs. (13) and (14) to double-check the accuracy of some of our numerical results. Formula (15) was used with the approximation that the series was truncated at $N=20$ terms. With $N=20$ reasonable accuracy was obtained for t small. The results were comparable to those obtained by other methods. We have not investigated the problem of choosing a suitable number N. Preliminary calculations indicated that N must increase with t to obtain accurate results.

The routine DERFC of Ref. 31 was used to obtain values of $R(t)$ using Eq. (16). Tests were made that gave results comparable with those obtained for $R(t)$ by solving the integral equation (14). These tests were for $p=p_4$ for the t-interval $0<t<0.3589\times10^{-4}$. For $t>0.8570\times10^{-4}$ the routine was abnormally terminated with the error message "t too big."

Figure 2 is a graph of the solution w_1 of Eq. (13) when $p=p_1$. Figure 3 is this same graph rescaled so that more detail of the later oscillations can be seen. Figures 4–6 show the solution of Eq. (13) for p_2, p_3, and p_4. Figure 7 is a rescaled version of Fig. 6. For all values of p the computed solution w_1 is a damped oscillation. As p increases the rate of oscillation increases, the maximum amplitude decreases, and damping increases. For example, for $p_1=0.9752$ the average distance between successive maxima and minima is about $T=0.0008$, the distance between successive zeros is also $T=0.0008$, the maximum attained height is $H=3310$, and the magnitude of any local maximum (or minimum) is about 23% ($d=0.23$) of the previous minimum (or maximum). The approximate values of these parameters are given in Table 2 for all four cases.

If these data are used to find parameters, then we see that over the range of the solutions it is approximately true that

$$
\begin{aligned}
w_1(t, p_1) &= 6800e^{-1800t}\sin(3900t)\\
w_1(t, p_2) &= 1800e^{-16,000t}\sin(39,000t)\\
w_1(t, p_3) &= 430e^{-99,000t}\sin(330,000t)\\
w_1(t, p_4) &= 110e^{-320,000t}\sin(1,750,000t)
\end{aligned}
\tag{18}
$$

As we shall see next, the behavior of these solutions is different than the behavior one would see with a viscous damping model.

Table 2 Modal periodicity, amplitude, and damping constants

p_j	T	H	d
0.975212	0.0008	3310	0.23
5.73930	0.00008	960	0.27
31.0359	0.000095	278	0.39
132.959	0.000018	86	0.56

Comparison with Kelvin-Voight

The Kelvin-Voight beam model with damping constant γ takes the form

$$\rho y_{tt} + 2\gamma \frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2 y}{\partial z^2 \partial t}\right] + \frac{\partial^2}{\partial z^2}\left[EI\frac{\partial^2 y}{\partial z^2}\right] = 0$$

If the Kelvin-Voight constitutive assumption is used in the control problem depicted in Fig. 1 and the same approximation using Eq. (4) is applied we arrive at the same transfer function $C(h^2)$ given in Eq. (5). However, instead of Eq. (2) it will now be the case that

$$h^2 = s^2/(1+2\gamma s)$$

Hence a typical term in Eq. (5) will have the form

$$\frac{1}{h^2+p_j^2} = \frac{1+2\gamma s}{s^2+2\gamma p_j^2 s+p_j^2} \tag{19}$$

The inverse Laplace transform of a term (19) will be oscillatory whenever $\gamma<p^{-1}$. If $\gamma<p^{-1}$ then the inverse transform of Eq. (19) will be

$$w_1(t, p_j) = 2|A|e^{-\gamma p_j^2 t}\sin\left(p\sqrt{1-(\gamma p_j)^2}\, t+\tau\right)$$

where

$$A = \gamma + i\frac{1-2\gamma^2 p_j^2}{2p_j\sqrt{1-(\gamma p_j)^2}}, \qquad i^2=-1$$

and where $e^{i\tau}=iA/|A|$. If γ is small, then $A\cong\gamma+i/2p_j$ and

$$w_1(t, p_j) \cong (1/p_j)e^{-\gamma p_j^2 t}\sin(p_j t), \qquad j=1, 2, 3, 4$$

Such solutions are not even roughly compatible with Eq. (18). By this we mean that the sine terms in Eq. (18) do not correspond to $\sin(p_i t)$ for any value p_i near those in Table 1. Similarly, for the given values of p_i, there is no value of $\gamma>0$ such that the exponential terms in Eq. (18) are roughly of the form $e^{-\gamma p_i^2 t}$. (Indeed for $j=1$ one computes $\gamma\cong1800$, for $j=2$ $\gamma\cong490$, for $j=3$ $\gamma\cong100$, and for $j=4$ $\gamma\cong18$.) Similarly the constant terms in Eq. (18) are 6800, 1800, 430, and 110 whereas Kelvin-Voight predicts constants of the approximate values 1, 0.17, 0.03, and 0.007.

We conclude that Kelvin-Voight damping predicts a time domain response that is very different from that of the fractional derivative model.

Solution of the Control Problem

The zero state angular response $\theta(t)$ of the feedback system in Fig. 1 (with $\theta_d=1$) takes the form

$$\theta(t) = \sum_{j=1}^{4}\beta_j W_j(t) \tag{20}$$

where W_j is defined by

$$W_j(t) = w_2(t, p_j) = \int_0^t w_1(s, p_j)\,ds$$

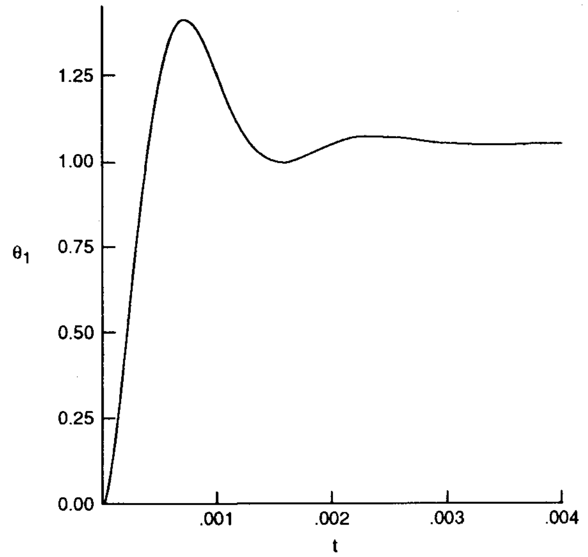

Fig. 8 $w_1(t) = w_2(t, p_1)$.

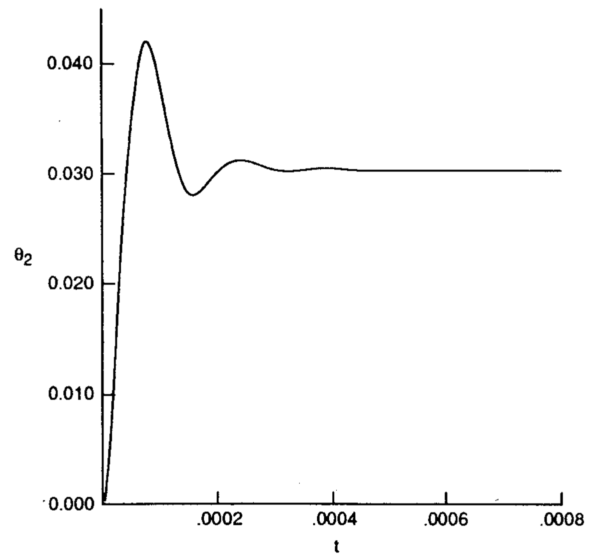

Fig. 9 $w_2(t) = w_2(t, p_2)$.

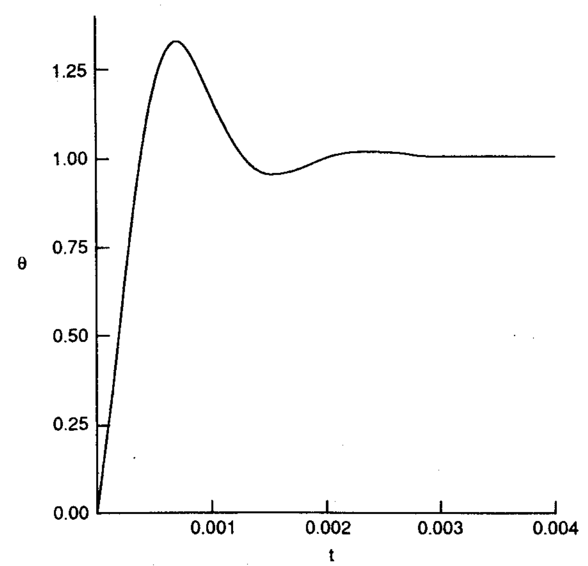

Fig. 10 Two-term approximation of θ.

For large values of t the integral $\beta_j W_j(t)$ will contribute approximately β_j/p_j^2 to $\theta(t)$, i.e., 0.94966, 0.04775, 0.00251, and 0.00008. These numbers suggest only two terms are needed for a reasonable approximation to $\theta(t)$. The suggestion turns out to be correct. The graphs of $W_1(t)$ and $W_2(t)$ are given in Figs. 8 and 9. The two-term approximation

$$\theta(t) \cong \beta_1 W_1(t) + b_2 W_2(t)$$

is graphed in Fig. 10. This computation predicts rise time of about 0.00034, an overshoot of about 33% followed by a 5% undershoot. Thereafter the solution damps very rapidly to its limiting value of one. One might guess that the rise time, overshoot, and undershoot could be varied with different choices of the gain K^*. Our numerical procedures would allow full exploration of this question. This could be done by recalculating $\theta(t) = \theta(t, K^*)$ for different values of K^* and seeing how $\theta(t, K^*)$ changes with K^*.

Conclusion

We studied a control strategy for a one-dimensional, distributed, viscoelastic system whose constitutive law is modeled using fractional derivatives. We considered a model problem consisting of a beam rotating on a rigid hub with a feedback pointing control on the hub. We showed that the transfer function and the zero state response can be successfully approximated. The transfer function for this fractional derivative damping model is very different from the same transfer function under Kelvin-Voight assumptions. Performance information can easily be obtained for various gains. Our techniques can be generalized to a wide range of problems well beyond the scope of our simple example, e.g., to a variety of damped systems including systems with a mix of viscoelastic and rigid elements.

Acknowledgments

Athena Makroglou gratefully acknowledges a computing grant award from the Iowa State University Computation Center, Account I4090. Richard K. Miller's work is supported by the National Science Foundation under Grant DMS-8901574. Steven Skaar's work is supported by the Office of Naval Research under ONR Contract N00014-91-J-1054. Numerical results were obtained on a DEC Station 5000 computer. A number of test runs were also made on the HDS AS/9180 computer at Iowa State University. We wish to thank James A. Hoekstra from the High Performance Computing Group, Computation Center, Iowa State University for his assistance with programming the HDS AS/9180.

References

[1]Skaar, S. B., Michel, A. N., and Miller, R. K., "Stability of Viscoelastic Control Systems," *IEEE Transactions,* Vol. AC-33, No. 4, 1988, pp. 348–357.

[2]Bagley, R. L., and Calico, R. A., "Fractional Order State Equations for the Control of Viscoelastic Damped Structures," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 2, 1991, pp. 304–311.

[3]Bagley, R. L., and Torvik, P. J., "Fractional Calculus—A Different Approach to the Analysis of Viscoelastically Damped Structures," *AIAA Journal,* Vol. 21, No. 5, 1983, pp. 741–748.

[4]Bagley, R. L., and Torvik, P. J., "On the Appearance of the Fractional Derivative in the Behavior of Real Materials," *Journal of Applied Mechanics,* Vol. 51, June 1988, pp. 294–298.

[5]Bagley, R. L., and Torvik, P. J., "Fractional Calculus in the Transient Analysis of Viscoelastically Damped Structures," *AIAA Journal,* Vol. 23, No. 6, 1985, pp. 918–925.

[6]Rogers, L., "Damping: On Modeling Viscoelastic Behavior," *Shock and Vibrations Bulletin,* Vol. 51, May 1981, pp. 55–59.

[7]Doetsch, G., *Guide to the Application of the Laplace and Z Transforms,* Van Nostrand Reinhold, London, 1971.

[8]Doetsch, G., *Introduction to the Theory and Application of the Laplace Transformation,* Springer-Verlag, Berlin, 1937.

[9]*Handbook of Mathematical Functions,* National Bureau of Standards Applied Mathematics Series, edited by M. Abramowitz and

I. A. Stegen, Vol. 55, U.S. Government Printing Office, Washington, DC, 1964.

[10]Davies, B., and Martin, B., "Numerical Inversion of the Laplace Transform: A Survey and Comparison of Methods," *Journal of Computational Physics,* Vol. 33, No. 1, 1979, pp. 1-32.

[11]Piessens, R., and Dang, N. P. D., "A Bibliography on Numerical Inversion of the Laplace Transform and Applications: A Supplement," *Journal of Computational Applied Mathematics,* Vol. 2, No. 3, 1976, pp. 225-228.

[12]Ang, D. D., Lund, J., and Stenger, F., "Complex Variable and Regularization Methods of Inversion of the Laplace Transform," *Mathematics of Computation,* Vol. 53, No. 188, 1989, pp. 589-608.

[13]Garbow, B. S., Giunta, G., Lyness, J. N., and Murli, A., "Algorithm 662: A FORTRAN Software Package for the Numerical Inversion of the Laplace Transform Based on Weeks Method," *ACM Transactions on Mathmatical Software,* Vol. 14, No. 2, 1988, pp. 171, 172.

[14]Honig, G., and Hirdes, U. "Alg. 27: A Method for the Numerical Inversion of Laplace Transforms," *Journal of Computational and Applied Mathematics,* Vol. 10, No. 1, 1984, pp. 113-132.

[15]Piessens, R., and Huysmans, R., "Algorithm 619, Automatic Numerical Inversion of the Laplace Transform," *ACM Transactions on Mathematical Software,* Vol. 10, No. 3, 1984, pp. 348-353.

[16]Gripenberg, G., London, S. O., and Staffans, O., *Volterra Integral and Functional Equations,* Cambridge University Press, Cambridge, England, UK, 1990.

[17]Miller, R. K., *Nonlinear Volterra Integral Equation,* W. A. Benjamin, Inc., Menlo Park, CA, 1971.

[18]Baker, C. T. H., *The Numerical Treatment of Integral Equations,* Clarendon Press, Oxford, England, UK, 1977.

[19]Brunner, H., and van der Houwen, P. J., *The Numerical Solution of Volterra Equations,* North Holland, Amsterdam, The Netherlands, 1986.

[20]Linz, P., *Analytical and Numerical Methods for Volterra Equations,* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1985.

[21]Caputo, M., and Mainardi, F., "Linear Models of Dissipation in Anelastic Solids," *Rivista Del Nuovo Cimento,* Vol. 1, No. 2, 1971, pp 161-198 (see esp. p. 196).

[22]De Hoog, F., and Weiss, R., "Implicit Runge-Kutta Methods for Second Kind Volterra Integral Equations," *Numerische Mathematik,* Vol. 23, No. 3, 1975, pp. 199-213.

[23]Makroglou, A., "Numerical Solution of Volterra Integro-Differential Equations," Ph.D. Thesis, Dept. of Mathematics, Univ. of Manchester, Manchester, England, UK, 1977.

[24]Makaroglou, A., "Convergence of a Block-by-Block Method for Nonlinear Volterra Integro-Differential Equations," *Mathematics of Computation,* Vol. 35, No. 151, 1980, pp. 783-796.

[25]Brunner, H., "Implicit Runge-Kutta Methods of Optimal Order for Volterra Integrodifferential Equations," *Mathematics of Computation,* Vol. 42, No. 165, 1984, pp. 95-109.

[26]Makroglou, A., "A Block-by-Block Method for Volterra Integro-Differential Equations with Weakly-Singular Kernel," *Mathematics of Computation,* Vol. 37, No. 155, 1981, pp. 95-99.

[27]Makroglou, A., "Product Integration Methods with Nonequidistant Knots for Volterra Integro-Differential Equations with Weakly Singular Kernels," unpublished manuscript, 1983.

[28]Makroglou, A., "A Block-by-Block Method for Volterra Delay Integro-Differential Equations with Weakly-Singular Kernel Combined with Product Integration on Nonuniform Meshes," 12th Biennial Conference on Numerical Analysis, Dundee, Scotland, June 1987.

[29]Brunner, H., "The Numerical Solution of Weakly Singular Volterra Integral Equations by Collocation on Graded Meshes," *Mathematics of Computation,* Vol. 45, No. 172, 1985, pp. 417-437.

[30]Brunner, H., "Polynomial Spline Collocation Methods for Volterra Integrodifferential Equations with Weakly Singular Kernels," *IMA Journal of Numerical Analysis,* Vol. 6, 1986, pp. 221-239.

[31]Kahaner, D., Moller, C., and Nash, S., "Numerical Methods and Software," Prentice-Hall, Englewood Cliffs, NJ, 1989.